IX

The Schwarz Lemma and Hyperbolic Geometry

This short chapter is devoted to the Schwarz lemma, which is a simple consequence of the power series expansion and the maximum principle. The Schwarz lemma is proved in Section 1, and it is used in Section 2 to determine the conformal self-maps of the unit disk. In Section 2 we formulate the Schwarz lemma to be invariant under the conformal self-maps of the unit disk, thereby obtaining Pick's lemma. This leads in Section 3 to the hyperbolic metric and hyperbolic geometry of the unit disk.

1. The Schwarz Lemma

The Schwarz lemma is easy to prove, yet it has far-reaching consequences.

Theorem (Schwarz Lemma). *Let $f(z)$ be analytic for $|z| < 1$. Suppose $|f(z)| \leq 1$ for all $|z| < 1$, and $f(0) = 0$. Then*

$$|f(z)| \leq |z|, \qquad |z| < 1. \tag{1.1}$$

Further, if equality holds in (1.1) *at some point $z_0 \neq 0$, then $f(z) = \lambda z$ for some constant λ of unit modulus.*

For the proof, we factor $f(z) = zg(z)$, where $g(z)$ is analytic, and we apply the maximum principle to $g(z)$. Let $r < 1$. If $|z| = r$, then $|g(z)| = |f(z)|/r \leq 1/r$. By the maximum principle, $|g(z)| \leq 1/r$ for all z satisfying $|z| \leq r$. If we let $r \to 1$, we obtain $|g(z)| \leq 1$ for all $|z| < 1$. This yields (1.1). If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = 1$, and by the strict maximum principle, $g(z)$ is constant, say $g(z) = \lambda$. Then $f(z) = \lambda z$.

An analogous estimate holds in any disk. If $f(z)$ is analytic for $|z - z_0| < R$, $|f(z)| \leq M$, and $f(z_0) = 0$, then

$$|f(z)| \leq \frac{M}{R}|z - z_0|, \qquad |z - z_0| < R, \tag{1.2}$$

with equality only when $f(z)$ is a multiple of $z - z_0$. This can be proved directly, based on the factorization $f(z) = (z - z_0)g(z)$. It can also be obtained from (1.1) by scaling in both the z-variable and the w-variable, $w = f(z)$, and by translating the center of the disk to z_0, as follows. The change of variable $\zeta \mapsto R\zeta + z_0$ maps the unit disk $\{|\zeta| < 1\}$ onto the disk $\{|z - z_0| < R\}$. If we define $h(\zeta) = f(R\zeta + z_0)/M$, then $h(\zeta)$ is analytic on the open unit disk and satisfies $|h(\zeta)| \le 1$ and $h(0) = 0$. The estimate $|h(\zeta)| \le |\zeta|$ becomes (1.2).

The Schwarz lemma gives an explicit estimate for the "modulus of continuity" of an analytic function. It shows that a uniformly bounded family of analytic functions is "equicontinuous" at each point. We will return in Chapter XI to treat the ideas of equicontinuity and compactness for families of analytic functions.

There is an infinitesimal version of the Schwarz lemma.

Theorem. *Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \le 1$ for $|z| < 1$, and $f(0) = 0$, then*

$$|f'(0)| \le 1, \tag{1.3}$$

with equality if and only if $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.

The estimate (1.3) follows by taking $z \to 0$ in the Schwarz lemma. For the case of equality, we consider the factorization $f(z) = zg(z)$ used in the proof of the Schwarz lemma, and we observe that $g(0) = f'(0)$. If $|f'(0)| = 1$, we then have $|g(0)| = 1$, and we conclude as before from the strict maximum principle that $g(z)$ is constant. Hence $f(z) = \lambda z$.

Note that the estimate (1.3) is the same as the Cauchy estimate for $f'(0)$ derived in Section IV.4, without the hypothesis that $f(0) = 0$. See also Exercise 7.

Exercises for IX.1

1. Let $f(z)$ be analytic and satisfy $|f(z)| \le M$ for $|z - z_0| < R$. Show that if $f(z)$ has a zero of order m at z_0, then
$$|f(z)| \le \frac{M}{R^m} |z - z_0|^m, \qquad |z - z_0| < R.$$
Show that equality holds at some point $z \ne z_0$ only when $f(z)$ is a constant multiple of $(z - z_0)^m$.

2. Suppose that $f(z)$ is analytic and satisfies $|f(z)| \le 1$ for $|z| < 1$. Show that if $f(z)$ has a zero of order m at z_0, then $|z_0|^m \ge |f(0)|$. *Hint.* Let $\psi(z) = (z - z_0)/(1 - \overline{z_0}z)$, which is a fractional linear transformation mapping the unit disk onto itself, and show that $|f(z)| \le |\psi(z)|^m$.

3. Suppose that $f(z)$ is analytic for $|z| \leq 1$, and suppose that $1 < |f(z)| < M$ for $|z| = 1$, while $f(0) = 1$. Show that $f(z)$ has a zero in the unit disk, and that any such zero z_0 satisfies $|z_0| > 1/M$. *Hint.* For the second assertion, consider $\psi(f(z))$, where $\psi(w)$ is a fractional linear transformation mapping 1 to 0 and the circle $\{|w| = M\}$ to the unit circle. Or use Exercise 2.

4. Suppose that $f(z)$ is analytic for $|z| < 1$ and satisfies $f(0) = 0$ and $\operatorname{Re} f(z) < 1$. (a) Show that $|f(z)| \leq 2|z|/(1 - |z|)$. *Hint.* Consider the composition of $f(z)$ and the fractional linear transformation mapping the half-plane $\{\operatorname{Re} w < 1\}$ onto the unit disk. (b) Show that $|f'(0)| \leq 2$. (c) For fixed z_0 with $0 < |z_0| < 1$, determine for which functions $f(z)$ there is equality in (a). (d) Determine for which functions $f(z)$ there is equality in (b). (e) By scaling the estimates in (a) and (b), obtain sharp estimates for $|g(z)|$ and $|g'(0)|$ for functions $g(z)$ analytic for $|z| < R$ and satisfying $g(0) = 0$ and $\operatorname{Re} g(z) < C$.

5. Suppose that $f(z)$ is analytic and satisfies $|f(z)| \leq 1$ for $|z| < 1$. Show that if $|f(0)| \geq r$, then $|f(z)| \geq (r - |z|)/(1 - r|z|)$ for $|z| < r$. Determine for which functions $f(z)$ equality holds at some point z_0 with $|z_0| < r$.

6. Let $f(z)$ be a conformal map of the open unit disk onto a domain D. Show that the distance from $f(0)$ to the boundary of D is estimated by $\operatorname{dist}(f(0), \partial D) \leq |f'(0)|$.

7. Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic for $|z| < 1$ and satisfies $|f(z)| \leq M$.
   (a) Show that $\sum_{k=0}^{\infty} |a_k|^2 \leq M^2$. *Hint.* Integrate $|f(z)|^2$ around a circle of radius r.
   (b) Show using (a) that $|f'(0)| \leq M$, with equality only if $f(z)$ is a constant multiple of z. *Remark.* It is not assumed that $f(0) = 0$.
   (c) Show that $|f^{(k)}(0)| \leq k!M$, with equality only if $f(z)$ is a constant multiple of z^k.

8. Suppose that $f(z)$ is analytic for $|z| < 1$ and satisfies $|f(z)| < 1$, $f(0) = 0$, and $|f'(0)| < 1$. Let $r < 1$. Show that there is a constant $c < 1$ such that $|f(z)| \leq c|z|$ for $|z| \leq r$. Show that the nth iterate $f_n(z) = f(f(\cdots f(z) \cdots)) = f(f_{n-1}(z))$ of $f(z)$ satisfies $|f_n(z)| \leq c^n|z|$ for $|z| \leq r$. Deduce that $f_n(z)$ converges to zero normally on the open unit disk $\mathbb{D}$.

2. Conformal Self-Maps of the Unit Disk

We denote by $\mathbb{D}$ the open unit disk in the complex plane, $\mathbb{D} = \{|z| < 1\}$. A **conformal self-map of the unit disk** is an analytic function from $\mathbb{D}$ to itself that is one-to-one and onto. The composition of two conformal self-maps is again a conformal self-map, and the inverse of a conformal self-map is a conformal self-map. The conformal self-maps form what is called a "group," with composition as the group operation. The group identity is the identity map $g(z) = z$.

For fixed angle φ, the rotation $z \mapsto e^{i\varphi}z$ is a conformal self-map of $\mathbb{D}$ that fixes the origin, and these are the only conformal self-maps that leave 0 fixed.

Lemma. *If $g(z)$ is a conformal self-map of the unit disk $\mathbb{D}$ such that $g(0) = 0$, then $g(z)$ is a rotation, that is, $g(z) = e^{i\varphi}z$ for some fixed φ, $0 \le \varphi \le 2\pi$.*

To see this, we apply the Schwarz lemma to $g(z)$ and to its inverse. Since $g(0) = 0$ and $|g(z)| < 1$, the Schwarz lemma applies, and $|g(z)| \le |z|$. If we apply the Schwarz lemma also to $g^{-1}(w)$, we obtain $|g^{-1}(w)| \le |w|$, which for $w = g(z)$ becomes $|z| \le |g(z)|$. Thus $|g(z)| = |z|$. Since $g(z)/z$ has constant modulus, it is constant. Hence $g(z) = \lambda z$ for a unimodular constant λ.

Theorem. *The conformal self-maps of the open unit disk $\mathbb{D}$ are precisely the fractional linear transformations of the form*

$$f(z) \;=\; e^{i\varphi}\frac{z-a}{1-\bar{a}z}, \qquad |z| < 1, \tag{2.1}$$

where a is complex, $|a| < 1$, and $0 \le \varphi \le 2\pi$.

Define $g(z) = (z-a)/(1-\bar{a}z)$. Since $g(z)$ is a fractional linear transformation, it is a conformal self-map of the extended complex plane, and it maps circles to circles. From

$$|e^{i\theta} - a| \;=\; |e^{-i\theta} - \bar{a}| \;=\; |1 - \bar{a}e^{i\theta}|, \qquad 0 \le \theta \le 2\pi,$$

we see that $|g(z)| = 1$ for $z = e^{i\theta}$, so that $g(z)$ maps the unit circle to itself. Since $g(a) = 0$, $g(z)$ must map the open unit disk to itself. Consequently, $g(z)$ is a conformal self-map of the unit disk, and so is $f(z)$ defined by (2.1). Let $h(z)$ be an arbitrary conformal self-map of $\mathbb{D}$, and set $a = h^{-1}(0)$. Then $h \circ g^{-1}$ is a conformal self-map of $\mathbb{D}$, and $(h \circ g^{-1})(0) = h(a) = 0$. By the lemma, $(h \circ g^{-1})(w) = e^{i\varphi}w$ for some fixed φ, $0 \le \varphi \le 2\pi$. Writing $w = g(z)$, we obtain $h(z) = e^{i\varphi}g(z)$, and $h(z)$ has the form (2.1).

The parameters a and $e^{i\varphi}$ are uniquely determined by the conformal self-map $f(z)$ of $\mathbb{D}$. The parameter a is $f^{-1}(0)$, and since

$$f'(z) \;=\; e^{i\varphi}\frac{1-|a|^2}{(1-\bar{a}z)^2}\,, \qquad |z|<1,$$

the parameter φ is uniquely specified (modulo 2π) as the argument of $f'(0)$. Thus there is a one-to-one correspondence between points of the parameter space $\mathbb{D}\times\partial\mathbb{D}$ and conformal self-maps of the open unit disk.

Next we take a giant step by proving a form of the Schwarz lemma that is invariant under conformal self-maps of the open unit disk.

Theorem (Pick's Lemma). *If $f(z)$ is analytic and satisfies $|f(z)|<1$ for $|z|<1$, then*

$$|f'(z)| \;\le\; \frac{1-|f(z)|^2}{1-|z|^2}\,, \qquad |z|<1. \tag{2.2}$$

If $f(z)$ is a conformal self-map of $\mathbb{D}$, then equality holds in (2.2)*; otherwise, there is strict inequality for all $|z|<1$.*

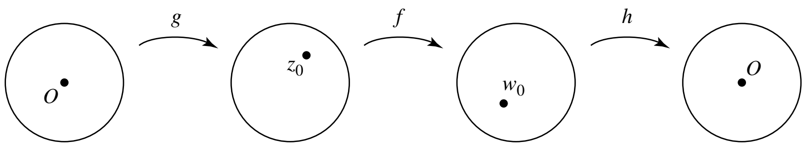
To prove (2.2), our strategy is to transport z and $f(z)$ to 0 using conformal self-maps, and to apply the Schwarz lemma to the resulting composition. Fix $z_0\in\mathbb{D}$ and set $w_0=f(z_0)$. Let $g(z)$ and $h(z)$ be conformal self-maps of $\mathbb{D}$ mapping 0 to z_0 and w_0 to 0, respectively, say

$$g(z) \;=\; \frac{z+z_0}{1+\overline{z_0}z}\,, \qquad\qquad h(w) \;=\; \frac{w-w_0}{1-\overline{w_0}w}\,.$$

Then $h\circ f\circ g$ maps 0 to 0. The estimate (1.3) and the chain rule yield

$$|(h\circ f\circ g)'(0)| \;=\; |h'(w_0)f'(z_0)g'(0)| \;\le\; 1, \tag{2.3}$$

hence $|f'(z_0)| \le 1/|g'(0)||h'(w_0)|$. Substituting $g'(0) = 1-|z_0|^2$ and $h'(w_0)=1/(1-|w_0|^2)$, we obtain (2.2).



If $f(z)$ is a conformal self-map of $\mathbb{D}$, then so is $h\circ f\circ g$, so we have equality in (2.3), which yields equality in (2.2). Conversely, suppose that $f(z)$ is an analytic function from $\mathbb{D}$ to $\mathbb{D}$ such that equality holds in (2.2) at one point z_0. Then the calculations above give $|(h\circ f\circ g)'(0)|=1$. According to Section 1, $h\circ f\circ g$ is multiplication by a unimodular constant, hence a conformal self-map of $\mathbb{D}$. Composing by h^{-1} on the left and by g^{-1} on the right, we conclude that f is a conformal self-map of $\mathbb{D}$.

Exercises for IX.2

1. A **finite Blaschke product** is a rational function of the form
$$B(z) \;=\; e^{i\varphi}\left(\frac{z-a_1}{1-\overline{a_1}z}\right)\cdots\left(\frac{z-a_n}{1-\overline{a_n}z}\right),$$
where $a_1, \dots, a_n \in \mathbb{D}$ and $0 \le \varphi \le 2\pi$. Show that if $f(z)$ is continuous for $|z| \le 1$ and analytic for $|z| < 1$, and if $|f(z)| = 1$ for $|z| = 1$, then $f(z)$ is a finite Blaschke product.

2. Show that $f(z) = (3+z^2)/(1+3z^2)$ is a finite Blaschke product.

3. Suppose $f(z)$ is analytic for $|z| < 3$. If $|f(z)| \le 1$, and $f(\pm i) = f(\pm 1) = 0$, what is the maximum value of $|f(0)|$? For which functions is the maximum attained?

4. For fixed $z_0, z_1 \in \mathbb{D}$, find the maximum value of $|f(z_1) - f(z_0)|$ among all analytic functions $f(z)$ on the open unit disk $\mathbb{D}$ satisfying $|f(z)| < 1$. Determine for which such functions the maximum value is attained. *Hint.* Consider first the case where $z_0 = r > 0$ and $z_1 = -r$, and show that the maximum is $2r$, attained only for $f(z) = \lambda z$, $|\lambda| = 1$.

5. Show that any conformal self-map of the upper half-plane has the form
$$f(z) = \frac{az+b}{cz+d}, \qquad \operatorname{Im} z > 0,$$
where a, b, c, d are real numbers satisfying $ad - bc = 1$. When do two such coefficient choices for a, b, c, d determine the same conformal self-map of the upper half-plane?

6. Show that the conformal maps of the upper half-plane onto the open unit disk are of the form
$$f(z) = e^{i\varphi}\frac{z-a}{z-\bar{a}}, \qquad \operatorname{Im} a > 0,\ 0 \le \varphi \le 2\pi.$$
Show that a and $e^{i\varphi}$ are uniquely determined by the conformal map.

7. Show that every conformal self-map of the complex plane $\mathbb{C}$ has the form $f(z) = az + b$, where $a \ne 0$. *Hint.* The isolated singularity of $f(z)$ at ∞ must be a simple pole.

8. Show that every conformal self-map of the Riemann sphere $\mathbb{C}^*$ is given by a fractional linear transformation.

9. Show that any conformal self-map of the punctured unit disk $\{0 < |z| < 1\}$ is a rotation $z \mapsto e^{i\varphi}z$.

10. Show that any conformal self-map of the punctured complex plane $\{0 < |z| < \infty\}$ is either a multiplication $z \mapsto az$, or such a multiplication followed by the inversion $z \mapsto 1/z$.

11. Let $D = \mathbb{C}\backslash\{a_1, \dots, a_m\}$ be the complex plane with m punctures. Show that any conformal self-map of D is a fractional linear transformation that permutes $\{a_1, \dots, a_m, \infty\}$.

12. Determine the conformal self-maps of the following domains D: (a) $D = \mathbb{C}\backslash\{0, 1\}$, (b) $D = \mathbb{C}\backslash\{-1, 0, 1\}$, (c) $D = \mathbb{C}\backslash\{-1, 0, 2\}$.

13. Suppose $f(z)$ is an analytic function from the open unit disk $\mathbb{D}$ to itself that is not the identity map z. Show that $f(z)$ has at most one fixed point in $\mathbb{D}$. *Hint.* Make a change of variable with a conformal self-map of $\mathbb{D}$ to place the fixed point at 0.

14. Suppose $f(z)$ is an analytic function from the open unit disk $\mathbb{D}$ to itself that is not a conformal self-map, and denote by $f_n(z)$ the nth iterate of $f(z)$. Show that if $f(z)$ has a fixed point $z_0 \in \mathbb{D}$, then $f_n(z)$ converges to z_0 for each $z \in \mathbb{D}$. Show that for each $r < 1$, the convergence is uniform for $|z| \le r$. *Hint.* See Exercise 1.8.

15. We say that two conformal self-maps f and g of $\mathbb{D}$ are **conjugate** if there is a conformal self-map h of $\mathbb{D}$ such that $g = h \circ f \circ h^{-1}$. (See the exercises for Section II.7.) Let f be a conformal self-map of $\mathbb{D}$ that is not the identity map z. (a) Show that either f has two fixed points on $\partial\mathbb{D}$, counting multiplicity, or f has one fixed point in $\mathbb{D}$. (b) Show that f has a fixed point in $\mathbb{D}$ if and only if f is conjugate to a rotation $g(z) = e^{i\varphi}z$. (c) Show that rotations by different angles are not conjugate. (d) Show that f has two distinct fixed points on $\partial\mathbb{D}$ if and only if f is conjugate to $g(z) = (z - s)/(1 - sz)$ for some s satisfying $0 < s < 1$. (e) Show that g's for different s's are not conjugate. (f) Show that any two conformal self-maps of $\mathbb{D}$ with one fixed point on $\partial\mathbb{D}$ (of multiplicity two) are conjugate.

3. Hyperbolic Geometry

Suppose $w = f(z)$ is a conformal self-map of the open unit disk $\mathbb{D}$. From Pick's lemma we then have equality in (2.2),

$$\left|\frac{dw}{dz}\right| = \frac{1 - |w|^2}{1 - |z|^2}.$$

In differential form this becomes

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2},$$

which means that if γ is any smooth curve in $\mathbb{D}$, and $w = f(z)$ is a conformal self-map of $\mathbb{D}$, then

$$\int_{f\circ\gamma} \frac{|dw|}{1-|w|^2} \;=\; \int_\gamma \frac{|dz|}{1-|z|^2}\,. \tag{3.1}$$

Thus to obtain a length function that is invariant under conformal self-maps of $\mathbb{D}$, we are led to make the following definition. We define the **length of γ in the hyperbolic metric** by

$$\text{hyperbolic length of } \gamma \;=\; 2\int_\gamma \frac{|dz|}{1-|z|^2}\,. \tag{3.2}$$

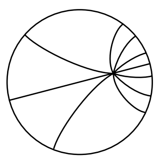
The factor 2 is a harmless factor, which is often omitted. (It adjusts the metric so that its curvature is -1.) The identity (3.1) shows that $f\circ\gamma$ has the same hyperbolic length as γ for any conformal self-map $f(z)$ of $\mathbb{D}$. Thus hyperbolic lengths are invariant under conformal self-maps of $\mathbb{D}$.

We define the **hyperbolic distance** $\rho(z_0, z_1)$ from z_0 to z_1 to be the infimum (greatest lower bound) of the hyperbolic lengths of all piecewise smooth curves in $\mathbb{D}$ from z_0 to z_1. Since conformal self-maps of $\mathbb{D}$ preserve the hyperbolic lengths of curves, they also preserve hyperbolic distances; that is, for any conformal self-map $w = f(z)$ of $\mathbb{D}$,

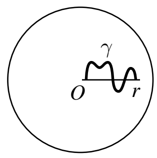
$$\rho(f(z_0), f(z_1)) \;=\; \rho(z_0, z_1), \qquad z_0, z_1 \in \mathbb{D}.$$

Theorem. *For any two distinct points z_0, z_1 in the open unit disk $\mathbb{D}$, there is a unique shortest curve in $\mathbb{D}$ from z_0 to z_1 in the hyperbolic metric, namely, the arc of the circle passing through z_0 and z_1 that is orthogonal to the unit circle.*

The paths of shortest hyperbolic length between points are called **hyperbolic geodesics**. The hyperbolic geodesics play the role that straight lines play in the Euclidean geometry of the plane. They satisfy all the axioms of Euclidean geometry except the parallel axiom (that through each point not on a given line there passes a unique straight line through the point and parallel to the given line).



hyperbolic geodesics



For a proof of the theorem, let $w = f(z)$ be a conformal self-map of $\mathbb{D}$ such that $f(z_0) = 0$. By multiplying by a unimodular constant, we can arrange that $f(z_1) = r > 0$. Since $f(z)$ preserves hyperbolic lengths, and since $f(z)$ maps circles orthogonal to the unit circle onto circles orthogonal to the unit circle, it suffices to show that the straight line segment from 0 to r is a unique path of shortest hyperbolic length from 0 to r. For this, let $\gamma(t) = x(t) + iy(t)$, $0 \leq t \leq 1$, be a piecewise smooth path in $\mathbb{D}$ from 0 to r. Then $\alpha(t) = \mathrm{Re}(\gamma(t)) = x(t)$ defines a path in $\mathbb{D}$ from 0 to r along the real axis, and

$$\int_\alpha \frac{|dz|}{1-|z|^2} \;=\; \int_0^1 \frac{|dx(t)|}{1-x(t)^2} \;\leq\; \int_0^1 \frac{|dx(t)|}{1-|\gamma(t)|^2} \;\leq\; \int_\gamma \frac{|dz|}{1-|z|^2}\,.$$

If $y(t) \neq 0$ for some t, then $|\gamma(t)| > |x(t)|$, and the first inequality above is strict. In this case, the path $\alpha(t)$ on the real axis is strictly shorter than the path $\gamma(t)$. Further, if $\alpha(t)$ is decreasing on some interval, we could reduce the integral by deleting a parameter interval over which $\alpha(t)$ starts and ends at the same value. We conclude that the integral is a minimum exactly when $\gamma(t)$ is real and nondecreasing, in which case the path is the straight line segment from 0 to r.

We turn now to an important reinterpretation of Pick's lemma.

Theorem. *Every analytic function $w = f(z)$ from the open unit disk $\mathbb{D}$ to itself is a contraction mapping with respect to the hyperbolic metric ρ,*

$$\rho(f(z_0), f(z_1)) \;\leq\; \rho(z_0, z_1), \qquad z_0, z_1 \in \mathbb{D}. \tag{3.3}$$

Further, there is strict inequality for all points $z_0, z_1 \in \mathbb{D}$, $z_0 \neq z_1$, unless $f(z)$ is a conformal self-map of $\mathbb{D}$, in which case there is equality for all $z_0, z_1 \in \mathbb{D}$.

To see this, let γ be the geodesic from z_0 to z_1. Then $f \circ \gamma$ is a curve from $f(z_0)$ to $f(z_1)$. Pick's lemma and the definition of the hyperbolic metric yield

$$\begin{aligned}\rho(f(z_0), f(z_1)) \;&\leq\; 2\int_{f\circ\gamma} \frac{|dw|}{1-|w|^2} \;=\; 2\int_\gamma \frac{|f'(z)|\,|dz|}{1-|f(z)|^2} \\ &\leq\; 2\int_\gamma \frac{|dz|}{1-|z|^2} \;=\; \rho(z_0, z_1).\end{aligned}$$

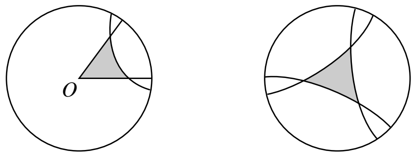
If $f(z)$ is not a conformal self-map of $\mathbb{D}$, there is strict inequality in Pick's lemma, and we obtain strict inequality in this estimate, hence in (3.3).

The hyperbolic distance from 0 to z can be computed explicitly. It is

$$\rho(0, z) \;=\; 2\int_0^{|z|} \frac{dt}{1-t^2} \;=\; \int_0^{|z|} \left[\frac{1}{1-t} + \frac{1}{1+t}\right] dt \;=\; \log\left(\frac{1+|z|}{1-|z|}\right).$$

This shows that the hyperbolic distance from 0 to z tends to $+\infty$ when z tends to the boundary of the unit disk.

A **geodesic triangle** is an area bounded by three hyperbolic geodesics. Since the hyperbolic geodesics and the angles between them are preserved



hyperbolic triangles

by conformal self-maps of $\mathbb{D}$, we can map any geodesic triangle to a triangle with vertex at 0 and with the same angles between sides. For a geodesic triangle with vertex at 0, two of the sides are radial segments, and the third is an arc of a circle lying inside the Euclidean triangle with the two radii as sides. From this representation we see that the sum of the angles of any geodesic triangle is strictly less than π, which is the sum of the angles of the corresponding Euclidean triangle.

In connection with complex analysis, we have now been in contact with three spaces with strikingly different geometries. The first space is the complex plane $\mathbb{C}$ with the usual Euclidean metric $|dz|$. In the Euclidean plane, the geodesics are straight lines, and the sum of the angles of a geodesic triangle is exactly equal to π. The second space is the open unit disk $\mathbb{D}$ with the hyperbolic metric $2|dz|/(1-|z|^2)$. For the hyperbolic disk, the geodesics are arcs of circles orthogonal to the unit circle, and the sum of angles of a geodesic triangle is strictly less than π.

The third space is the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ with the spherical metric, which can be introduced in a manner completely analogous to the hyperbolic metric. Recall (Section I.3) that the chordal metric induced on $\mathbb{C}$ by the Euclidean metric of the sphere via the stereographic projection is given explicitly by

$$\text{chordal distance from } z \text{ to } w \;=\; \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}.$$

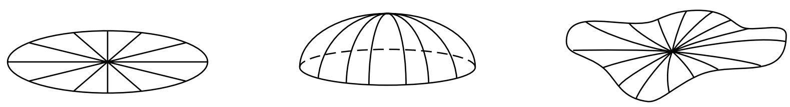
The infinitesimal form of this metric is $2|dz|/(1+|z|^2)$. If γ is a path in $\mathbb{C}^*$, its **length in the spherical metric** is

$$\text{spherical length of } \gamma \;=\; 2\int_\gamma \frac{|dz|}{1+|z|^2} \;=\; 2\int \frac{|\gamma'(t)|}{1+|\gamma(t)|^2}\,dt\,.$$

This is the length of the corresponding path on the unit sphere in $\mathbb{R}^3$. The **distance from z_1 to z_2 in the spherical metric** is defined to be the infimum of the spherical lengths of the paths joining z_1 to z_2. Since the chordal metric is invariant under rotations of the sphere, so is the spherical metric, and consequently, the lengths of paths and the distances between points in the spherical metric are invariant under rotations. It is not difficult to show that the geodesics in the spherical metric correspond to great circles on the sphere, and the sum of the angles of a geodesic triangle is strictly greater than π.

Each of these three spaces is homogeneous, in the sense that any prescribed point can be transported to any other by an "isometry." Thus for each of these spaces, any scalar quantity that is invariant under isometries is constant. It turns out that a notion of "scalar curvature" can be associated to each of the spaces (see the exercises), and the curvature is invariant under isometries, so that in each case the curvature is constant. The Euclidean plane has constant zero curvature, the sphere has constant positive curvature, and the hyperbolic disk has constant negative curvature. The curvature can be related to the area and the sum of angles of geodesic triangles (Gauss-Bonnet formula). We summarize these properties in tabular form.

Geometry	Euclidean	Spherical	Hyperbolic
Infinitesimal length	$\lvert dz\rvert$	$\dfrac{2\lvert dz\rvert}{1+\lvert z\rvert^2}$	$\dfrac{2\lvert dz\rvert}{1-\lvert z\rvert^2}$
Oriented isometries	$e^{i\varphi}z+b$	rotations	conformal self-maps
Curvature	0	$+1$	-1
Geodesics	lines	great circles	circles $\perp$ unit circle
Angles of triangle	$=\pi$	$>\pi$	$<\pi$
Disk circumference	$2\pi\rho$	$2\pi\rho-\dfrac{\pi\rho^3}{3}+\mathcal{O}(\rho^5)$	$2\pi\rho+\dfrac{\pi\rho^3}{3}+\mathcal{O}(\rho^5)$



Euclidean disk spherical disk hyperbolic disk

Exercises for IX.3

1. Show by direct computation that $|w'(z)| = (1-|w|^2)/(1-|z|^2)$ for any conformal self-map $w = f(z)$ of $\mathbb{D}$.

2. A **hyperbolic disk** centered at $z_0 \in \mathbb{D}$ of radius $\rho > 0$ consists of all $z \in \mathbb{D}$ such that $\rho(z, z_0) < \rho$. (a) Show that the hyperbolic disk centered at 0 of radius ρ is a Euclidean disk of radius $r = (e^\rho - 1)/(e^\rho + 1)$. (b) Show that any hyperbolic disk is a Euclidean disk.

3. Denote by $c(z,\rho)$ and $r(z,\rho)$ the Euclidean center and Euclidean radius of the hyperbolic disk centered at z of hyperbolic radius ρ. (a) For fixed ρ, show that $r(z,\rho)/(1-|z|)$ tends to a constant $A > 0$ as $|z| \to 1$. (b) For fixed ρ, show that $|z - c(z,\rho)|/r(z,\rho)$ tends to a constant B, $0 < B < 1$, as $|z| \to 1$.

4. Show that the circumference of a hyperbolic disk of radius ρ is $2\pi \sinh \rho$. *Hint.* Show first that the hyperbolic circumference of a Euclidean disk of radius r centered at 0 is $4\pi r/(1-r^2)$.

5. We define the **hyperbolic area** of a subset E of $\mathbb{D}$ to be

$$4 \iint_E \frac{dx\, dy}{(1-|z|^2)^2}.$$

Show that the hyperbolic area is invariant under conformal self-maps of $\mathbb{D}$. Show that the hyperbolic area of a hyperbolic disk of radius ρ is given by

$$2\pi(\cosh \rho \;-\; 1) \;=\; \pi\rho^2 + \frac{\pi}{12}\rho^4 + \mathcal{O}(\rho^6).$$

6. Establish the following, for the spherical metric. (a) The circumference of a spherical disk of radius ρ is $2\pi \sin \rho$, $0 < \rho < \pi$. (b) The area of a spherical disk of radius ρ is given by

$$2\pi(1-\cos\rho) \;=\; \pi\rho^2 - \frac{\pi}{12}\rho^4 + \mathcal{O}(\rho^6).$$

(c) The geodesics in the spherical metric correspond to great circles on the sphere. *Hint.* It suffices to show that the shortest curve from 0 to ε in the spherical metric is the straight line segment joining them.

7. Show that an isometry of the hyperbolic disk $\mathbb{D}$ is either a conformal self-map of $\mathbb{D}$ or the composition of a conformal self-map and the reflection $z \mapsto \bar{z}$.

8. Let $f(z) = (az+b)/(cz+d)$, where $ad-bc = 1$. Show that $f(z)$ is an isometry in the spherical metric if and only if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary.

9. Show that the function $f(z) = z^2$ is strictly contracting with respect to the hyperbolic metric on any subdisk $\{|z| \le r\}$, $0 < r < 1$, and that any branch of the square root function is strictly expanding, by establishing the following. (a) For fixed r, $0 < r < 1$, show that

$$\rho(z^2, \zeta^2) \;\le\; \frac{2r}{1+r^2}\rho(z,\zeta), \qquad |z|, |\zeta| \le r.$$

When does equality hold? (b) Show that the constant $2r/(1+r^2)$ in (a) is sharp. (c) For fixed s, $0 < s < 1$, show that

$$\rho\left(\pm\sqrt{z}, \pm\sqrt{\zeta}\right) \;\ge\; \frac{1+s}{2\sqrt{s}}\rho(z,\zeta), \qquad |z|, |\zeta| \le s.$$

10. Show that

$$d(z,w) \;=\; \left|\frac{z-w}{1-\bar{w}z}\right|, \qquad |z|,|w|<1,$$

satisfies the triangle inequality, that is, $d(z,w) \leq d(z,\zeta)+d(\zeta,w)$ for all $z,\zeta,w \in \mathbb{D}$. *Remark.* This can be regarded as the analogue of the chordal metric for the sphere (defined in Section I.3). Except for the constant factor 2, the hyperbolic metric is the infinitesimal version of the metric function $d(z,w)$.

11. Show that the metric function $d(z,w)$ defined in the preceding exercise satisfies

$$d(f(z),f(w)) \leq d(z,w), \qquad |z|,|w|<1,$$

for any analytic function $f(z)$ from $\mathbb{D}$ to $\mathbb{D}$. Show that equality obtains whenever $f(z)$ is a conformal self-map of $\mathbb{D}$, and otherwise there is strict inequality for all $z \neq w$.

12. A conformal map $g(z)$ of a domain D onto the open unit disk $\mathbb{D}$ induces the metric ρ_D on D defined by

$$d\rho_D(z) = \frac{2|g'(z)|}{1-|g(z)|^2}|dz|\,, \qquad z \in D.$$

Show that ρ_D is independent of the conformal map $g(z)$ of D onto $\mathbb{D}$. *Remark.* The metric ρ_D is called the **hyperbolic metric** of the simply connected domain D.

13. Show that the hyperbolic metric of the upper half-plane $\mathbb{H}$ is given by

$$d\rho_{\mathbb{H}}(z) = \frac{|dz|}{y}, \qquad z = x+iy,\ y>0.$$

What are the geodesics in the hyperbolic metric? Illustrate with a sketch.

14. Show that the horizontal strip $S = \{-\pi/2 < \operatorname{Im} z < \pi/2\}$ has hyperbolic metric

$$d\rho_S(z) = \frac{|dz|}{\cos y}, \qquad z = x+iy,\ -\pi/2<y<\pi/2.$$

Sketch the hyperbolic geodesics that are orthogonal to the vertical interval $\{iy : -\pi/2 < y < \pi/2\}$.

15. The **curvature** of the metric $\sigma(z)|dz|$ is defined to be

$$\kappa(z) \;=\; -\frac{1}{\sigma(z)^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\sigma(z).$$

Find the curvature of each of the spherical, the hyperbolic, and the Euclidean metrics.

16. (Wolff-Denjoy Theorem.) Let $f(z)$ be an analytic function from $\mathbb{D}$ to $\mathbb{D}$. Let $f_n(z)$ denote the nth iterate of $f(z)$, and let K_r denote the closed disk $\{|z| \leq r\}$.
    (a) Show that if $f(z)$ is not a conformal self-map of $\mathbb{D}$, then for any $r < 1$ there is a constant $c < 1$ such that $\rho(f(z), f(w)) \leq c\rho(z, w)$ for $z, w \in K_r$.
    (b) Show that if the image $f(\mathbb{D})$ is contained in K_r for some $r < 1$, then the iterates $f_n(z)$ converge uniformly on $\mathbb{D}$ to a fixed point for $f(z)$.
    (c) Show that if $f(z)$ is not a conformal self-map of $\mathbb{D}$, and if there is $r < 1$ such that the iterates of some point $z_0 \in \mathbb{D}$ visit K_r infinitely often, then the iterates $f_n(z)$ converge normally on $\mathbb{D}$ to a fixed point of $f(z)$. *Hint.* First find the fixed point.
    (d) Show that if the iterates of some point $z_0 \in \mathbb{D}$ tend to the unit circle $\partial\mathbb{D}$, then there is a point $\zeta \in \partial\mathbb{D}$ (the **Wolff-Denjoy point**) such that the iterates $f_n(z)$ converge normally on $\mathbb{D}$ to ζ. *Hint.* Suppose $z_0 = 0$. Define $g_\varepsilon(z) = (1 - \varepsilon)f(z)$, let z_ε be the fixed point of $g_\varepsilon(z)$, and let D_ε be the hyperbolic disk centered at z_ε with 0 on its boundary. Show that the limit D of the D_ε's is a Euclidean disk that is invariant under $f(z)$ and whose boundary meets $\partial\mathbb{D}$ in exactly one point.

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